

ORDER OF THE WORLD OR ORDER OF THE SOCIAL – A WITTGENSTEINIAN CONCEPTION OF MATHEMATICS AND ITS IMPORTANCE FOR RESEARCH IN MATHEMATICS EDUCATION

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In this article the connection between the philosophy of mathematics and mathematics education is discussed. Special focus is on the implications of different conceptions of the nature and importance of mathematics. The argument will be made that the later Wittgenstein presents us with an unreservedly social interpretation of mathematics that favours a certain direction for our research on mathematics education. According to this interpretation, mathematics could be considered to be constituted exclusively in complex social processes, in which case any conception of it mirroring a pre-existing world of mathematical objects is rejected. To contrast with the Wittgensteinian position, a Platonist position is presented and the two philosophical positions are discussed in relation to their significance for mathematics education.

INTRODUCTION

In the 20th century, many philosophers of mathematics turned their focus on highly technical discussions on the foundations of mathematics, thereby indirectly closing the field down for engaging in close cooperation with other fields of research. These foundational discussions may have been fruitful from a mathematical point of view but a side effect has been the absence of a place in which philosophical questions about much broader issues relating to mathematics could be discussed. Interest in understanding and thinking about the role played by mathematics in society from a philosophical perspective seems to have suffered, as have continued reflections on the relation between the philosophy of mathematics and mathematics education.

Several authors have, however, continuously discussed such interconnections in the overlapping space between mathematics education and the philosophy of mathematics and made important contributions to our conception of this connection (see for example Ernest (1993)). Recently there has been a related interest in the connections between mathematics education, the philosophy of mathematics, and the sociology of mathematics (Kerkhove & Bendegem, 2007), as well as important research on the use of philosophical ideas on mathematics in planning mathematics curricula (François & Bendegem, 2007).

In this article, we shall attempt a similar exploration into the connections between the philosophy of mathematics and mathematics education. Here, we shall be especially concerned with the question of the nature and importance of mathematics and its significance for thinking about mathematics education.

Considerations about the nature and importance of mathematics define a particular space of connection between the two research fields. In this space, central questions for both researchers in mathematics education and the philosophy of mathematics can be posed: “What role does mathematics play in society?”; “Why is something at this level of abstraction given such a prominent position among the sciences?”; “Why is mathematics considered to be so important that it holds a central position at all levels of curricula all around the world?” These are just some of the questions one could imagine to be relevant for both philosophers of mathematics and mathematics education researchers.

The focus of this article will be two partial investigations into this space of research. Here we shall attempt to investigate a) How Wittgenstein’s conception of mathematics presents to us a purely social interpretation of mathematics and thereby brings about the best possible foundation for an analysis of the social nature of and role played by mathematics in society, and b) How considerable differences in philosophical standpoints on the nature of mathematics could influence our research in mathematics education.

In connection with the first enquiry, we shall develop an in-depth description of Wittgenstein’s conception of mathematics and attempt to explain how it presents to us the most profoundly social account of mathematics imaginable. In the second enquiry we shall investigate the implications of interpreting the nature of mathematics, in turn, from a Wittgensteinian perspective and then an opposing Platonist conception.

It seems to me that there are many good reasons to bring a Wittgensteinian perspective to the fore. First of all, Wittgenstein is thought by many to be the most inventive philosopher of the 20th century, and even his opponents often acknowledge that his position is one of those that must be contested in a variety of research areas if one wants to successfully make different claims to his (for an example from the philosophy of mathematics, see Katz (1998)). Another reason is that he offers a new approach to understanding the social nature of mathematics that differs from the one expounded by for example Imre Lakatos, in his otherwise groundbreaking considerations about the quasi-empirical nature of mathematical objects and the social processes involved in mathematical discovery (Lakatos, 1981). Thereby Wittgenstein seems to me an important figure in discussions on the nature and significance of mathematics in society that aim to clarify and widen our insight into the social role played by mathematics.

In the following section, we start out by contrasting the Wittgensteinian position by considering an entirely contrary idea about the nature and importance of mathematics, to which we shall refer as a Platonist conception. In this conception mathematics is part of the fundamental structure of the world – uninfluenced by the doings of human beings – and only through acquiring knowledge of mathematics is it possible to lay bare the fundamental characteristics of the world. After briefly

outlining the main ideas of this Platonist conception in the philosophy of mathematics, we shall consider at some length the Wittgensteinian ideas about what characterises mathematics.

In the final section of the paper, it will be discussed how these two divergent philosophical interpretations of mathematics present to us different visions for thinking about mathematics education and why we should pay a special interest to Wittgenstein's findings on the nature and importance of mathematics in society.

MATHEMATICS AS THE ORDER OF THE WORLD

In the Western history of philosophical thought about mathematics, one position surpasses all others as a sort of crest or initial theory as to what mathematics is all about. I am of course referring to Plato's position, and it is necessary for the task we have set ourselves here to clarify what is meant by a Platonist conception of mathematics. We shall develop such key features of a Platonist conception, not by directly studying what Plato himself has said on the subject, but instead primarily by looking at what is normally referred to as Platonism in the contemporary philosophy of mathematics. In addition we shall briefly look at some of the historical roots and the persistence of this conception of mathematics.

We shall concentrate on three aspects of a Platonist philosophy of mathematics – that people who delve into mathematics are discovering facts about an already existing mathematical reality; that this mathematical realm is an extremely well-structured entity and finally the idea that mathematics should be considered a hidden but very important order of the world.

Pre-existing mathematics

The conception of mathematics connected with Plato's philosophy is often referred to as *Platonism*. Platonism, in the widest sense of the word, refers to the idea that there exists a mathematical reality that mathematical theories seek to uncover. Instead of Platonism, therefore, discussions often concern the slightly broader school of *realism*. Mathematics is assumed to have a genuine field of objects. Like physics analyses the physical nature, biology analyses plants and animals, and geology analyses rock formations, so mathematics analyses geometrical shapes, numbers, and whatever else that belongs among mathematical objects in a similar fashion.

This 'in a similar fashion' should be considered in more depth, because while physics, biology, and geology can employ empirical methods, the objects of mathematics are not accessible to our senses. They exist outside of time and space, which makes them empirically inaccessible. But humankind possesses a capability other than the senses for understanding, namely reason. When cultivated in its sublime form, this capability gives access to the objects of mathematics. Thus Platonism claims the existence of an eternal mathematical world of objects and holds

that our reason can reveal truths about this world. These truths are then also eternal and necessary, thanks to the immutability of mathematical objects.

This may sound rather prodigious, but let us look at Platonism in a more everyday version by way of an example. It has been proven, and the proof can be found in Euclid among others, that there are an infinite number of primes. It is not proven, however, that there are infinitely many prime twins, i.e. ordered pairs of the form 3-5, 5-7, 11-13, 17-19, ..., 450797-450799, ... etc. We know that the density of primes decreases up through the scale.[1] And when primes are farther and farther apart, it is conceivable that the occurrence of prime twins will cease altogether at some point. The statement that there are infinitely many prime twins is, however, neither proved nor disproved.

How does this problem look from the point of view of Platonism? The sentence “there are infinitely many prime twins” has to be either true or false. It states something about mathematical reality. Mathematics has simply been unsuccessful so far in mapping out this particular part of mathematical reality. Such altogether sensible formulations represent a Platonist way of thinking about mathematics.

Platonist and non-Platonist perspectives will have implications for the way mathematical activities are viewed and interpreted. A Platonist will see such activity as an exploration of a hitherto unknown world, discovering more and more truths about an already existing – eternal and immutable – mathematical world. Another perspective, to which we will return later, holds mathematics to be a human construction. We ‘build up’ mathematics from the ground. Both perspectives seem to have something of importance to say about mathematics but this also poses a dilemma in our thinking about mathematics. In *What is Mathematics, Really?* (1997) Ruben Hersh points out that it is perfectly normal for mathematicians to be Platonists on weekdays but not during the weekend. By this he means that mathematicians go about their work practice as though they were uncovering truths about a mathematical reality. It is only in more detached moments, e.g. when they are asked about their work, that many mathematicians distance themselves from the somewhat peculiar contention that mathematical objects which we can neither see nor touch should exist in reality. Nevertheless it is certainly no exaggeration that Platonism is quite a common perspective today, among mathematicians as well as other people.

The Grand Structure of Mathematics

While Platonism in this general sense is expressive of a comparatively ‘simple’ idea about the reality of mathematics, Plato’s own Platonism is somewhat more complex and we shall omit a thorough look into Plato’s own version of Platonism here. It is however not without reason that he should hold such a high position in the philosophy of mathematics. Plato lived from 427 to 347 BC, much of that time in Athens, and he had many disciples after founding the school that he named “the Academy”. [2] Above the entrance of the Academy was inscribed, “Let none ignorant

of geometry enter here”, and this headline was in many ways exemplary of the importance Plato ascribed to the mathematical training of his students.

At the Academy Plato’s students and other thinkers laid some of the groundwork for Euclid’s *Elements* in the following century and thereby nurtured the idea that mathematics is a world of unity – an entity of truths which can be represented as an ordered entity. It has been speculated that Euclid must have attended Plato’s academy for him to have achieved such profound insight into, among other things, the mathematical work of Eudoxus and Theaitetos.

Euclid (c. 300 BC) is considered the pioneer with regard to the axiomatisation of mathematics. In Euclid’s *Elements* geometry is decisively presented as a unified entity. Euclid managed to join together much of the known mathematics at the time into one grand structure of interconnected theorems and proofs that rested on only five basic axioms in addition to the definitions of the basic geometric concepts like ‘point’, ‘line’ and ‘plane’. The five axioms stood for thousands of years as the foundation of mathematics.[3]

These axioms are followed by a long chain of proofs and theorems leading to more proofs of more theorems etc. You only use the theorems that have already been acknowledged as true for the proof of new theorems all based on the truth of the axioms. This construction of interconnected theorems goes on for thirteen books starting with the first theorem dealing with equilateral triangles and finishing with theorems 13-17 in Book 13 of the *Elements* that deal with the regular polyhedra. There are only five of these regular polyhedra (among them the cube) and they are also known as the “Platonic bodies” as Plato refers to them as the elements and building stones of the world in his dialogue called *Timaeus*. There is no question that Euclid carefully chose to begin Book 1 with a theorem on regular figures and to finish the entire work of his mathematical structure with his theorems on spatial regular figures. In this way, his work is in line with the basic idea within a Platonist conception of mathematics, where mathematics is considered a fundamental structure of the world that human beings can explore through the gradual buildup of mathematical theorems from a few rational and self-evident axioms.

Hence, the idea of the axiomatisation of mathematics can be thought of as a natural part of a Platonist conception of mathematics. I have made this idea a focal point here because it is important to be aware that this is a particular way of representing mathematics that could be contested, as we shall see later on. It is also a very dominant and influential way of understanding mathematics with a very long history. Even though the Euclidean system only deals with geometry, it has been a paradigmatic example of how real mathematics is to be represented and presented which has had a deep influence on our thinking about mathematics. At the same time, the work of Euclid reveals to us how mathematics in the Platonist framework is considered in some way or another to be the order of the world – in Plato’s outline

the buildings blocks of the universe, namely the elements. This leads us on to the last focal point of the Platonist conception.

The Order of the World

We have considered two aspects of what we refer to here as a Platonist conception of mathematics – the pre-existing reality of mathematics and thereby its independence from human activities, and the idea that mathematics is a unified structure of knowledge. Let us finally add one more feature to the Platonist conception.

Much later in Western history, the Platonist conception of mathematics is still a force to be reckoned with, and as we reach the renaissance breakthrough of early modern science, the idea is formed that the divine construction of the universe is hidden and written in mathematics. In order to illustrate this continued adherence to a Platonist conception of the nature of mathematics, we could consider how Kepler (1571-1630) – one of the superstars of the breakthrough of modern science – was deeply inspired by Greek philosophy, and like the Pythagoreans and Platonists he had no doubt that a mathematical reasoning was needed to fathom the construction of the heavenly spheres of the universe. In his first principal work, *Mysterium Cosmographicum* (1596), the Pythagorean-Platonist influence is impossible to miss in his arguments for the number of planets. He favours the explanation that there is exactly six planets in the solar system because there are five regular polyhedra in addition to the sphere (Field, 1988).

Today one might claim that such ‘rationalistic’ explanations about the cosmos are long gone, but the role we attribute to mathematics in fathoming the world does not seem to have waned. Let us finish our outline of a Platonist conception of mathematics by rephrasing a famous quote of Galileo that seems to me to decisively explain the idea that mathematics is the fundamental logic of the world.

Philosophy is written in this grand book, the universe, which stands continually open to our gaze, but the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth. (Galileo cited in Crosby, 1997, p. 240).

According to Galileo the universe has been thought out, written down, and put in front of us – just like a book. And this book is not written in Hebrew or Aramaic – as many former generations might have held – but in mathematics. Mathematics, according to Galileo is not just afforded the power of an effective instrument in discovering the world, rather it is the way the world basically is. In this way we can trace a long history of interpreting mathematics as the logic of the world. Whether God was believed to have installed the mathematical edifice and left it for us to discover, or the faith in an architect God had gradually been left behind in modern

times, the idea that mathematics is the construction blueprint for the fundamental workings of the universe has never really been abandoned.

MATHEMATICS AS THE ORDER OF THE SOCIAL WORLD

If one is to present a contrast to the Platonist conception of mathematics, several 20th century philosophers come to mind. Brouwer, Hilbert, Dummett and many others have each in their way launched a critique of a Platonist conception of mathematics. Their ideas build on Kant's outline of mathematics and epistemology and in different ways contrast a Platonist conception of mathematics, be it from an intuitionist approach or a more constructivist approach. From inside the field of mathematics education Paul Ernest (1993) and Reuben Hersh (1997), have both argued at length about the social nature of mathematics. These writers in the space between the philosophy of mathematics and mathematics education have corroborated the theory that many of the historically important conceptions of mathematics are 'inhuman' in stressing the other-worldly character of mathematics and its independence from human interference. They conclude from detailed studies and building on Lakatos' ideas on mathematics that mathematics as a construction of knowledge invented by humans resembles the empirical sciences.

Social interactions do play an important part in these philosophies of mathematics, but they probably do not represent the full step towards conceiving of mathematics as a purely social enterprise – a point to which we shall return later on. It could of course be that it is an impossible, or even an unimaginable endeavour to describe mathematics as a purely social enterprise. If mathematics works more or less like physics and geology each with their own objects to study, it might be inconceivable to suggest an interpretation of mathematics that is built only on human social interactions.

As mentioned earlier, we shall be paying special attention to Ludwig Wittgenstein's (1889-1951) ideas about mathematics in what follows, as we have set forward the task to explore how his conception of mathematics is the most thoroughly social and human-centered explanation possible. Wittgenstein is a peculiar character in the philosophy of mathematics. His explicit writings on the nature and foundation of mathematics have been judged insignificant by many authors, on the account that Wittgenstein did not grasp the actual content of the foundational debate going on in the 1930s and 1940s. It has been said that Wittgenstein never completed his writings on mathematics, and that we only have fragments of a position which are insufficient to characterise his writings as an important contribution to the philosophy of mathematics (Shanker, 1986, p. 2). However, it is unthinkable that one of the greatest philosophers of last century should have nothing of importance to say about a topic on which he spent years of research.

Different issues do make it difficult to approach Wittgenstein's writings on mathematics. First of all, Wittgenstein's work does not present its audience with an

all-inclusive theory. Instead, small and more or less independent paragraphs consisting of thought experiments, problems, explanations and rhetorical questions comprise his texts. As Wittgenstein himself explains, he is not interested in doing the thinking for the reader, but to make the reader think (Wittgenstein, 1994, p. 32). This is an issue closely related to his proposition that certain of our basic convictions about mathematics be subjected to a thorough rethinking. Secondly, his thoughts on mathematics, as presented in the posthumously published *Remarks on the Foundation of Mathematics* (1935-36), are not easily derived from this text alone, as it draws on his general philosophy of language from his principal work, *Philosophical Investigations* (1936).

In this later period of his writing, it is fair to say that he developed a groundbreaking conception of mathematics that posed a challenge to all earlier theories on the nature of mathematics.[4] He tries to explain to his reader how our general conception of mathematics is in several ways misleading. He sets the task to let the “fly out of the bottle” that is to dissolve our entanglements in excessively metaphysical, philosophical and theoretical uses of our everyday practised use of mathematical symbols. In what follows we shall develop some of the basic concepts from Wittgenstein’s general philosophical vocabulary, before we attempt to understand how we might think radically different thoughts about mathematics compared to the Platonist approach.

Language-Games and Family Resemblances

Wittgenstein’s early philosophy is contained in his first major work *Tractatus Logico Philosophicus* from 1922, a work that attempts to illustrate how language, beneath the surface of our everyday use of it, has a strict logical structure. By his ‘later’ writings, one refers to the ideas he came to hold during his last years. More precisely, this is the period from 1929, when he started to be sceptical about the concept of language presented in *Tractatus*, a scepticism he held until his death in 1951 (Gefwert, 1998, pp. 7-8). In order to get a clear conception of the philosophy of mathematics promoted by Wittgenstein during this period, it is essential to understand the new notion of language that he presented.

Whereas his former principal work *Tractatus* expressed a formal theory with its system of propositions, his later principal work *Philosophical Investigations* is deliberately written in an informal style. The aim is no longer to give a formal theory of the way in which we can uniquely express ourselves through language, but rather to emphasise the complexity of ways in which our language functions. Therefore, *Philosophical Investigations* consists of small paragraphs in no strictly determined order. It is formed as a discussion between Wittgenstein and his (imaginary) opponent, who often expresses Wittgenstein’s earlier thinking from *Tractatus*. This style emphasises the later Wittgenstein’s thesis that everyday language is the foundation of meaningful use of language.

According to the theory on language and meaning presented in *Tractatus*, language supposedly works as a medium to depict relations between objects, and words gain their meaning through this reference to objects. In questioning this understanding of the correlation between words and objects, Wittgenstein sets the agenda for a rejection of theories of language that focus on the descriptive features of language. Instead, he favours the opinion that words and sentences gain their meaning from the contexts in which we make use of them. An example of this is the sentence “Can she walk?”. Uttered by an uncle it could mean whether or not his niece has taken her first steps, but asked to a doctor the meaning could be one of concern for a victim of an accident. Whereas in *Tractatus*, Wittgenstein had claimed that a sentence has a fixed meaning in the composition of its constituent parts, he now holds that there is no single, fixed meaning of a sentence. The circumstances under which a sentence is uttered – the use made of the sentence – is what fixes its meaning.

Wittgenstein therefore introduces the notion of ‘language-games’. It is meant to underline the fact that speaking a language is part of an activity or life form (Wittgenstein, 1997, p. 11e [23]). Language in its totality is also called ‘the language-game’ from time to time. Examples of language-games are to command and to act; to describe something; to talk about an event; to make jokes; to solve equations; and so on. Language works as a number of tools with which we can perform a vast variety of different actions. Wittgenstein shows this variety of forms in which we use language meaningfully and maintains that this is in sharp contrast to the traditional interpretation of meaning and language of which his earlier publication *Tractatus* is an example.

It is interesting to compare the multiplicity of the tools in language and of the ways they are used, the multiplicity of kinds of word and sentence, with what logicians have said about the structure of language. (Including the author of the *Tractatus Logico-Philosophicus*.) (Wittgenstein, 1997, p. 12e [23])

Wittgenstein does not try to give an explanation catching the ‘essence’ of language as he did earlier in *Tractatus*. On the contrary, he refutes any notion that there exists a common characteristic for the class of activities we call language. The phenomenon we call language consists of a multiplicity of uses and there is no essential common characteristic between these uses (Wittgenstein, 1997, p. 31e [65]). Instead Wittgenstein proposes that the meaning of words and sentences flow from their use in human practice, i.e., in our language-games, and hence he maintains that the meaning of a word does not have any real existence as a physical, mental or ideal object. The adequate approach to finding the meaning of a word is an analysis of the use of the word in the appropriate language-games. Therefore the main theme of Wittgenstein’s later writings has been expressed as follows: The meaning of a word is its use.

The relation between different language-games is indicated by the term ‘games’. The variety of activities we call games have no obligatory common characteristics. Some games include the use of a round ball; others include a board, and so on. The partial

similarities that might be between two or more language-games are what Wittgenstein calls ‘family resemblances’ (Wittgenstein, 1997, pp. 31e-32e [66-67]).

I cannot characterise these resemblances better than by the word “family resemblances”; for the various resemblances which exist between different members of a family: height, facial features, eye colour, walk, temper, etc. etc., overlap and cross each other in exactly this way. – And I would say: the ‘games’ make up a family. (My translation from Wittgenstein, 1994, p. 67 [67])

This concept reflects the lack of essence or a unique common feature in the activities we call games. All there is, is a complicated net of familiarities overlapping each other. With the concept of family resemblance, he refutes the idea of commonality among things that we categorise together (e.g. games or mathematics) and the idea that exact definitions of words are necessary in order for them to have meaning (e.g. of the word ‘game’).

On Wittgenstein’s account, the quest for definitions of essences is in vain. To understand a word of our language is simply to be able to use the word according to certain ‘rules’ attached to the language-games in which the word is embedded. The extension of a word has no exact limits. Wittgenstein does not reject that we can, to a certain degree, specify limits to the extension of a word, but his point is that a word’s lack of such limits has never worried us when we have used it in practice (Wittgenstein, 1997, pp. 32e-33e [68]).

Rule-following

The foregoing considerations will become clearer when we analyse the important notion of following a rule. Wittgenstein’s considerations on rule-following are meant as an elaboration of the ‘meaning-is-use’ conception of language. With these considerations, the aim is to explain that to use a language is to follow rules. With his remarks on rule-following, Wittgenstein tries to show us that the idea we normally attach to following a rule (and as we shall see for doing inferences in mathematics) is basically wrong. This is the idea of there being ‘bodies of meaning’ underlying and determining the use and extension of a word or a rule (Shanker, 1987, p. 16).

Along with this idea goes the thought that when someone has grasped the rule, it must be followed in a certain way. When we speak of following a rule, we think of some guidance, which we are to follow precisely in order to do things the right way. Wittgenstein rejects this idea, as he maintains that rules in themselves cannot explain to us how to follow them. Whether a rule is presented to us through a formula, a signpost or something else, it is always possible to interpret these signs differently from that which we call the correct way to follow them (Malcolm, 1986, p. 158). We simply apply them as we do as a consequence of practice. With endless practice through exemplars in the use of equations we finally become very certain of how to manipulate them. What logically compels us to follow the rule ‘add 2’ in the way we do is that following the shared rule is itself the criterion for understanding the rule. It

is we who, through our practice, determine what is to count as the correct way to follow the rule.

A most important aspect of Wittgenstein's comments on rule-following is the inability for a single individual to 'fix the meaning of a rule' (Malcolm, 1986, p. 156). For there to be a difference between following a rule and believing that one is following the rule, there has to be some external criteria by which this difference can be established. If a single individual were to try and fix the meaning of a rule, what they believed to be the correct application of the rule would never be challenged. But then nothing they could possibly do would ever count as a wrong application of the rule. There would be no difference between believing one followed the rule and actually following the rule. This ultimately leads to the rule losing its meaning, as there are no criteria for what defines the wrong and the right applications (Malcolm, 1986, p.156). Thus the activity of rule-following requires a community in relation to which it can be determined whether the rule in question is followed according to normal practice. Practice is therefore seen as the necessary condition for establishing meaning and rules, where a practice is understood as a community of rule-followers who have had the same kind of training and therefore agree on the implications of certain rules.

According to Wittgenstein, we can be absolutely certain of how to use a rule, but still not be able to give ultimate reasons for following the rules as we do (Wittgenstein, 1979, p. 39e [307]). To understand a rule is parallel to the understanding of a word discussed above. You do not *have* something in your mind or elsewhere when you understand a word or a rule; but rather you are able to *do* something. To understand a word or a rule is comparable to mastering a technique. You are simply able to use the rule or the word within a language-game in accordance with its established use. This agreement is the bedrock of our explanations, because it constitutes the possibility of language. Without this agreement there would be no rules. This agreement in 'doing the same' is an example of what Wittgenstein calls a 'form of life', which is what we must accept as 'the given' that escapes explanation (Wittgenstein, 1997, p. 226e).

The Language-Games of Mathematics

By introducing some of the basic concepts of Wittgenstein's philosophical vocabulary, we have already touched upon his conception of the nature of mathematics. According to Wittgenstein's interpretation of language, it would appear that mathematics is considered a network of different language-games that share family resemblances. According to Wittgenstein, mathematical objects (equations, functions etc.) do not stand for anything. Instead, and as a consequence of his conception of language, they acquire their meaning from the rules we attach to them and according to which we use them; how we calculate with them. By a 'calculation' Wittgenstein therefore means a certain procedure for manipulating mathematical objects, e.g. deriving one equation from another, according to certain rules (Wrigley,

1986 (II), p. 186). The nature of mathematics in Wittgenstein's terminology is therefore: 'Mathematics consists entirely of calculations' (Wrigley, 1986 (II), p. 186).

The calculation conception of mathematics is a radical and entirely new approach to the traditionally important question: What is the nature of mathematical propositions? Wittgenstein holds the view that it is wrong to think of mathematics as consisting of a body of propositions having a meaning in themselves, as this easily leads us to think that these propositions were somehow there before we constructed them. In other words, he emphasises that the nature of mathematics consists in different techniques of calculations, rather than a body of true propositions (Wittgenstein, 1978, p. 365 [VII-8]).

This means that Wittgenstein does not see the theorems of mathematics as self-explanatory. Wittgenstein says that a mathematical proposition is connected to its proof like the surface of a body is connected to the body itself. Thus, proof and proposition in mathematics are intrinsically tied together. If we have no proof for a certain mathematical proposition, as until recently was the case with respect to Fermat's Last Theorem, it is actually wrong for us to call it a proposition. A mathematical 'conjecture' is the appropriate term, as we have not yet established any rules to govern its use in the mathematics. With this distinction, Wittgenstein wants to show us that a mathematical conjecture is a stimulus for our constructions, and not a meaningful proposition in need of a proof to confirm its truth or falsity.

A mathematician is of course guided by associations, by certain analogies with the previous system. After all, I do not claim that it is wrong or illegitimate if anyone concerns himself with Fermat's Last Theorem. Not at all! If e.g. I have a method for looking at integers that satisfy the equation $x^2 + y^2 = z^2$, then the formula $x^n + y^n = z^n$ may stimulate me. I may let a formula stimulate me. Thus I shall say, here there is a stimulus – but not a question. Mathematical problems are always such stimuli. (Wittgenstein, taken from Shanker, 1987, p. 113)

The proof gives meaning to its resulting proposition, and it would be misguided to say that the proof has changed the meaning of the conjecture. The conjecture is meaningless, as it has no place within the meaningful frame of a language-game. It is a stimulus and not a question, because a question presupposes that we have a method for answering it. Fermat's Last Theorem was a stimulus to all of us, as we associated this conjecture with the case of $n = 2$. But only after Andrew Weyl's proof was it legitimate to talk about this theorem as a meaningful mathematical proposition, in Wittgenstein's interpretation.

Proofs and Experiments

The close connection between mathematical proof and mathematical proposition illustrates an important aspect of the language-games we call mathematics. Whereas the natural sciences have each their own type of objects on which to perform

experiments, mathematics is the practice where we lay down the grammatical rules for description in the sciences and many other language-games.

Let us remember that in mathematics we are convinced of grammatical propositions; so the expression, the result, of our being convinced is that we accept a rule.

I am trying to say something like this: even if the proved mathematical proposition seems to point to a reality outside itself, still it is only the expression of acceptance of a new measure (of reality). Thus we take the constructability (provability) of this symbol (that is, of the mathematical proposition) as sign that we are to transform symbols in such and such a way. (Wittgenstein, 1978, pp. 162-163 [III-26-27])

The construction of a proof convinces us of the proposition, but it does so in the normative sense of our accepting a new measure of reality. This acceptance determines what makes sense to say and what does not. Because it is a grammatical proposition, the mathematical proposition cannot be refuted by an experiment. It has nothing to do with empirical matters whatsoever. In contrast to empirical propositions, it makes no sense to doubt the mathematical proposition, as doubt has been excluded from it, because we use it as a grammatical rule. This means that mathematics and the natural sciences consist of language-games of completely different natures.

We feel that mathematics stands on a pedestal – this pedestal it has because of a particular role that its propositions play in our language games.

What is proved by a mathematical proof is set up as an internal relation and withdrawn from doubt. (Wittgenstein, 1978, p. 363 [VII-6])

Hence, Wittgenstein very strictly maintains a distinction between proofs in mathematics and experiments in the sciences. Proofs distinguish themselves by being just that type of technique from which doubt is logically excluded. The distinction between our use of the concepts ‘proof’ and ‘experiment’ therefore indicates that mathematics has an important characteristic that it does not share with the natural sciences. And Wittgenstein’s maintaining how very different the language-games of mathematics function compared to those of science also makes it clear that the perception of mathematics he presents to us is different from the social constructivist conception mentioned earlier on.

Mathematics as Conventional Measures

Wittgenstein’s account of the questions concerning the genesis and growth of mathematical knowledge is basically of a conventional nature. His explanation is based on our freedom to invent new rules of grammar to follow (Shanker, 1986, p. 21). Sometimes this consists in constructing new links between ‘old’ mathematical concepts, and sometimes it consists in the construction of entirely new mathematical systems. Mathematics continually forms new rules and extends the old network of mathematics (Wittgenstein, 1978, p. 99). The construction of complex numbers is just one example of this.

Wittgenstein holds that mathematical conventions are concerned with the creation of new systems of representations.

What I want to say is: mathematics as such is always measure, not thing measured.
(Wittgenstein, 1978, p. 201 [III-75])

Mathematical forms of representation provide us with standards for representation in our description of the world. They are measures in the sense that they set up the rules of grammar through which we can describe something. Wittgenstein exemplifies the role played by grammatical rules with Einstein's use of Bolyai-Lobatchevskian geometry in his Theory of Relativity. Einstein's use of this alternative geometry (as opposed to the use of Euclidean geometry) is seen as an application of an alternative system of mathematical rules that decides the grammar for describing phenomena (Shanker, 1987, p. 270). Mathematical systems are thus seen as different grids or structures by which we measure or describe the world. According to Wittgenstein, some rules of grammar are presupposed in any description of reality (Shanker, 1987, p. 318), and in the sciences, mathematics is presupposed in this way.

In connection to Wittgenstein's conventionalist account of mathematics, it seems obvious to ask whether the development of mathematics is arbitrary or not. Wittgenstein comments on this aspect of the development in the following passage.

But then doesn't it [mathematics] need a sanction for this? Can it extend the network arbitrarily? Well, I could say: a mathematician is always inventing new forms of description. Some, stimulated by practical needs, others, from aesthetic needs, - and yet others in a variety of ways. And here imagine a landscape gardener designing paths for the layout of a garden; it may well be that he draws them on a drawing-board merely as ornamental strips without the slightest thought of someone's sometime walking on them.
(Wittgenstein, 1978, p. 99 [167])

Wittgenstein's answer is that the development of mathematics is arbitrary in so far as there is nothing in reality which compels or necessitates us to develop mathematics as we do. In another sense, we are, however, always guided in developing new mathematics. An established tradition can guide our mathematical constructions, and the trial against 'the facts of nature' in the sciences often generates new forms of mathematical representations. Thus, we have reason to construct mathematics in certain directions, but this does not mean that these reasons must somehow be justified by a correlation between reality and mathematical forms of representation (Shanker, 1987, p. 319). The mathematical forms of representation are autonomous in the sense that their meaning consists in our use of the grammatical rules within the mathematical system. If 'facts of nature' or objectives we pursue suggest that we develop new forms of representation, these are not in any sense more true forms of representation, but simply new grammatical rules by which we can describe our surroundings.

Inference as Rule-following

To corroborate his view of mathematics, Wittgenstein knows that he has to convincingly explain the act of logical inference involved in mathematics as a natural part of his theory.

These considerations bring us up to the problem: In what sense is logic something sublime? (Wittgenstein, 1997, p. 42e [89])

Wittgenstein wants to show that there is no such thing as an ultra-experience of some sort of reality, to which inference must obey (Wittgenstein, 1978, p. 40 [8]). What he argues against in passages like this one is the conception of mathematics known from for example as Logicism and the logical positivists in general. Their position has also been called a conventionalist position, because they held that by convention we agree on certain basic self-evident truths in logic and mathematics from which all remaining truths can be tautologically derived. Many writers have conflated Wittgenstein's theory with the conventionalism of the logical positivists, but the very different conceptions of mathematical growth reveal the error of such interpretations (Shanker, 1986, p. 20). Wittgenstein, as opposed to the logical positivists, avoids the difficulty raised against conventionalist theories by Poincaré, who questioned how there could be new discoveries in mathematics if it consists entirely of tautologies. Mathematics has nothing to do with tautologies, in the later Wittgenstein's opinion, but rather one could say that mathematics is built up by conventions as opposed to tautologies. Wittgenstein would ask, from what source do these imminent consequences of the rules of inference stem in a conventionalist theory (Wrigley, 1986 (I), p. 362)?

On Wittgenstein's account, the meaning of symbols depends exclusively on the use we make of them. Nothing is hidden beneath the surface of our practice, and it is therefore misleading to think of mathematics as mechanically following self-evident rules, as did the logical positivists with their notion of mathematics as tautologies. Instead, Wittgenstein's conception of inference is an application of his thoughts on rule-following. On the process of inferring he says,

There is nothing occult about this process; it is a derivation of one sentence from another according to a rule; ... (Wittgenstein, 1978, p. 39 [I-6])

Inference takes place as a transformation of our expressions according to some paradigm (language-game) and the right way of performing this transformation is the accordance with a convention or use (Wittgenstein, 1978, p. 41 [I-9]).

Hence, the theory of rule-following is also applied to the mathematical practice; i.e., to the different techniques of calculation. This means that logical inference in mathematics simply amounts to sufficient practising within the accepted practice of mathematics. For example, the rule 'add 2' does not in itself explain how it is to be used. If one knows the meaning of the rule, one would know how to use it. But as

discussed earlier on, according to Wittgenstein, meaning is use; that is, understanding the rule is the capability of using the rule in accordance with mathematical practice.

It is through our mathematical practice that we determine what is to count as being in compliance with the rule in question. That mathematics is embedded in human practice means that the rules we learn to follow in mathematics are not of a kind which we can apply without thought, as logical positivists understood them. Machines can act in a rule-bound fashion if they are programmed to do so, but they cannot perform calculations, as they are not capable of justifying their application of the rule. Rule-following is about doing things for a reason, which is only possible for creatures having will, who can set up goals to pursue.

Rejection of Platonism

Wittgenstein's philosophy of mathematics can be seen as an attempt to make us abandon all aspects of the Platonist view of mathematics that has prevailed since the time of the ancient Greeks. Platonism holds that mathematical objects and relations between those exist independently of human practice and of humans' capability of discovering these mathematical facts. In the Platonist conception, there are bodies of meaning lurking around beneath the surface of the mathematical symbols used within the mathematical practice, and Wittgenstein would dismiss the possibility of such meaning that did not stem from our use of the symbols. His critique of the nature of logical inference as truth-preserving also stresses the fact that he sees no reason to believe that mathematical objects exist independently of us and have relations for us to discover.

The mathematician is an inventor, not a discoverer. (Wittgenstein, 1978, p. 99 [I-167])

Mathematics has often been considered to stem from worlds of ideas or basic intuitions etc., which has nothing to do with the actual practice of mathematics from which mathematical calculations gain their meaning. Wittgenstein therefore claims that not even natural numbers exist or refer to something independent of our language-games, as for example the inborn pre-linguistic mathematical intuitions proposed by Brouwer (Körner, 1986, p. 122). We must resist such unfounded but ever present temptation to idealise the words and numbers of our language.

Counting (and this means: counting like this) is a technique that is employed daily in the most various operations of our lives. And that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is inexorably insisted that we shall all say "two" after "one", "three" after "two" and so on. – But is this counting only a use, then; isn't there also some truth corresponding to this sequence?" The truth is that counting has proved to pay. – "Then do you want to say that 'being true' means: being usable (or useful)?" – No, not that; but that it can't be said of the series of natural numbers – any more than of our language – that it is true, but: that it is usable, and, above all, it is used. (Wittgenstein, 1978, pp. 37-38 [I-4])

From this quotation we see how the concept of truth can only be used within a language-game, and Wittgenstein emphasises that the things we actually do are the ultimate foundation of language as well as mathematics. Calculations are techniques embedded in human practice and hence dependent upon the teaching of mathematics and all the other connections it has to human life forms.

Hence, we see that the philosophical investigations that Wittgenstein performs with respect to mathematics do not consist of deriving the epistemological source of mathematical knowledge, as mathematics is a part of our life form and cannot be meaningfully talked about as true or false, absolute or fallible. Neither does he try to justify our mathematical knowledge. Instead, these investigations are an attempt to discern from the interpretations and metaphors that surround them the actual content of the different practices of calculations. Wittgenstein uses the terms 'prose' and 'calculation' to signify the philosophical problems of a linguistic nature attached to a mathematical practice and the actual activity in the practice, respectively (Gefwert, 1998, p. 236).

Let us here end this presentation of Wittgenstein's conception of mathematics with a summary of its most important features. Mathematics is considered a group of language-games that share family resemblances and has incorporated into them different types of calculations. The growth and genesis of mathematical knowledge stem from our freedom to construct new mathematical structures. These mathematical structures are forms of representations through which we can describe the world. Derivations within a mathematical structure are therefore certain by virtue of being grammatical constructions.

In this way Wittgenstein accomplishes a full-scale rearrangement of the Platonist conception of mathematics. Mathematics is a social practice, and not just in the 'normal' sense that people work together on mathematics; but in the sense that they establish the very meaning of mathematical objects and processes. Doing mathematics is a rule-following activity, but following these rules is not a mechanical action. Rather it is an exclusively human activity to interpret how a rule is to be applied under given circumstances.

THE IMPLICATIONS OF DIFFERENT PHILOSOPHICAL POSITIONS

Two investigations were outlined at the beginning of the paper, and by now we have completed the task of showing how Wittgenstein's conception of mathematics seems to be the most socially founded explanation about the nature and importance of mathematics imaginable. In addition we have contrasted Wittgenstein's position to a classical Platonist conception of mathematics and are now in a good position to examine what different visions they bring to the table for thinking about mathematics education.

The very different types of answers to the question about the nature of mathematics seem to me to implicate certain differences in the possible direction of our thinking

about mathematics education. Different types of visions for research agendas come to the fore, depending on the philosophical position one adheres to. In what follows I will attempt to discuss two themes – case studies if you like – within the field of mathematics education research, in order to highlight the differences that philosophical considerations about mathematics can effect in the standpoints taken on educational issues. These visions are not easily derived in a unilinear fashion but nonetheless, differing philosophical standpoints lead to different ways of talking about mathematics that could be of vast import in practice. Firstly we shall discuss these differences in relation to discussions on the learning processes of mathematics and subsequently to discussions on the content of mathematics education.

Learning mathematics

If there actually exists a mathematical world of truths before any human being has thought about mathematics, then mathematics education is concerned with bringing students to see the logical necessity they possess deep within them – or at least that is how Plato himself thought of learning mathematics, namely as recollection. We can get an idea about what he means by looking at the dialogue *Meno*, in which Socrates discusses virtues with Meno, who was a student of Gorgias, one of the worst sophists imaginable in Plato and Socrates' world. Halfway through the dialogue between Socrates and Meno, the former continues the dialogue with a slave boy who never learned mathematics, by way of example. In the course of the dialogue, the boy achieves some mathematical insight. Socrates sets the problem to find a square whose area is double that of a given square. The boy succeeds in solving the problem once Socrates has put him on the right track by way of rational dialogue.

In this way, Socrates demonstrates that the unskilled slave boy and hence all human beings have some ability to think rationally about mathematical ideas, and thus can be said to have innate ideas about the organisation of the world. According to Socrates, he had not offered the boy any information about mathematical coherences. He would only ask guiding questions. Thereby Socrates suggests that we have a foreknowledge of the correlation between mathematical ideas and objects, such as squares and lines. We only need to be reminded of this knowledge. The learning of mathematics is then merely a question of recollecting knowledge that we all already possess. And in order to recollect, we just need to think rationally and engage in dialogue about mathematical subjects.

The discussion of mathematics in *Meno* is just one example that illustrates how to think about a range of other circumstances in life, e.g. what is the good thing to do, what is just, and what is true about reality. It is no coincidence, however, that Plato employs mathematics to demonstrate these conditions regarding cognition. In the Platonist framework, mathematics is the discipline that affords insight into, and training to think rationally about the world of Ideas in general (here we recollect the words over the portal of his Academy).

For the Platonist another issue concerns the proper medium for the learning processes of mathematics. As mathematics is basically about a world of abstract objects, pure thinking is all that is needed to access this world. Drawings might be helpful tools in learning mathematics for the untrained mind – for example the slave boy – but they are always only secondary to the abstractions of the mind.

Although they make use of the visible forms and reason about them, they are thinking not of these, but of the ideals which they resemble; not of the figures which they draw, but of the absolute square and the absolute diameter, and so on [...] they are really seeking to behold the things themselves, which can only be seen with the eye of the mind? (Plato, *The Republic*, Book VII)

Drawings or imprecise communication about mathematics suffers from not meeting the criteria found to be the nature of the mathematical world of objects.

We may imagine a Wittgensteinian arguing against this line of reasoning. If there is nothing but human rule-following involved in doing mathematics, it becomes clear that only one thing really matters in learning mathematics, namely training within a community of practitioners doing mathematics. Mathematics has its foundation – just as the natural language – in our everyday use of the terms, signs etc. involved in the use of mathematics, in the numerous language-games that involve mathematics in one way or another. Nothing is hidden beneath the surface of the mathematical calculations of signs involved in the practised language games, and in this way there are no rational short cuts or speedy aha-experiences that will help gain an immediate insight into the world of mathematics. In the case of the slave boy, the Wittgensteinian could even argue that the reason why the slave boy is actually getting along quite well in discussing mathematical arguments is because in his everyday practices or language-games – concerning very worldly things – he has had a great deal of out-of-school-training in mathematics. Mathematics is part of his language-games and his life form.

On the second point – regarding the proper ways to represent mathematical arguments – the Wittgensteinian would hold that there are no metaphysical reasons for demanding that a ‘pure’ mathematical proof be as abstract as possible, and there are no reasons to claim that a graphical proof should be of a lesser nature than a thought experiment. Even an axiomatised line of proofs and theorems towards a certain mathematical proposition does not in principle take higher ground. Wittgenstein would argue that a Euclidean proof with all of its graphics or even a completely graphical argument is every bit as valid as any other means of proving mathematical propositions.

A consequence of the rationalistic conception favoured by the Platonist might be that it fosters an environment where it is more prestigious to be able to calculate in your mind than to use ‘techniques’ for calculating. If the logical deductions of mathematics are something sublime that you may or may not have an extraordinary insight into, then you had better be seen to manage it without the use of such earthly

measures as rough workings, drawings or helpful rules to guide your way. In Wittgenstein's interpretation of mathematics, this downplay of techniques in calculations and an excessive tribute shown to 'the eye of the mind' betrays the nature of mathematics.

What is the relevance of Socrates' account to contemporary mathematics education? We do see diverse degrees of aptitude for learning mathematics among students in every classroom around the world. In other words, there seems to be certain obstacles that may impede the learning of mathematics. How will the Platonist and Wittgensteinian account for this experience?

In a Platonist's conception of mathematics it would be perfectly sensible to suggest that not all humans alike have the same inborn skills required to learn mathematics at a highly abstract level. In *The Republic Book III*, in his discussions with Glaucon, Plato himself explains how different people are born with different abilities – some with a soul that is best suited for the work of the hand, some with a soul of gold best suited for thinking, and hence more prepared from the outset to learn mathematics and take political decisions. At times the centre of attention in mathematics education research has been the attributes of the individual learner. This research might be focused on the cognitive structures of the individual or the IQ of a student. It might be the case that IQ measures indicate that a student is unlikely to be successful in learning mathematics, and it might even be suggested that the capability of the individual is connected to the DNA support structure for doing abstract thinking.

To understand the Wittgensteinian vision about obstacles in learning mathematics, we must first of all divert our attention from the individual. Mathematics is the least individualised thing one can imagine, from a Wittgensteinian perspective, because it is necessarily a community of practitioners that settle on the meaning of the symbols used. The individual is always only the secondary bearer of mathematics. Primary for understanding educational practices is the socialisation into the language games of mathematics. These are interrelational human activities where the meanings of signs, operations, proofs, drawings etc. are determined and continuously negotiated. Singling out one individual's obstacles in learning mathematics is therefore in principle nothing to do with the difficulty of the subject of mathematics. No matter what the subject, training and continued use within a supporting community should in principle be enough for anybody to learn any conceivable mathematics as it is 'only' a game of rule-following.

Hence, the Wittgensteinian would think of the approach to learning mathematics and the obstacles associated with this process a purely social matter. What matters here is not some preset ordering of each individual's mind, the gene pool one has received or the like. Obstacles in learning mathematics is rather to be explained in terms of the social inclusion and exclusion processes at the micro-didactical level – for example in a class room – or explained through social background, which naturally has an effect on the preparedness and discipline a pupil or student is able to muster. In principle –

according to a Wittgensteinian conception of mathematics – there is nothing difficult or special about the topic of mathematics that should leave us baffled with the task of learning its many facets. In a sense it is just as easy or difficult as learning any other language, were it not for the ‘prose’ we attribute to the learning of mathematics. Relentless practice in a supporting environment is what it takes to be good at mathematics and perhaps eventually a mathematical ‘genius’. The social agenda of course sets its rigid limits to what any individual is likely to achieve in mathematical skills.

This line of reasoning about the learning processes of mathematics reveals the importance of understanding how the sociology of mathematics and sociological approaches in general can help to disclose who will be successful in learning mathematics. It opens up the research agenda for conceptualising why some learn and some do not, from a sociological point of view. It is the interaction between people in relation to mathematical learning – formal as well as informal – that will decide how well one is acculturated with the rules of the game. Here a distribution of power can be witnessed, where some are singled out as those who have the skills necessary and some as those who lack them. In the Wittgensteinian account, it takes a safe environment for years and years – not a sharp mind – to learn mathematics for the uninitiated; an environment where, for example, making mistakes is allowed, motivation is present etc.

The content of mathematics education

Let us turn to another field of consideration within mathematics education. The two different philosophies of mathematics discussed seem to be at odds when discussions turn to the content of mathematics.

What would be a rational content for the Platonist to suggest for educational purposes? For the Platonist the foundation of mathematics, its axiomatic foundation, the logical progress from natural numbers to rational numbers and so forth could be an organising principle. The world of mathematics is organised, and the more organised the content for education is presented, the easier it should be for the student to realise and clearly see the world of mathematics. This could mean, for example, that some fundamental operations should necessarily be learned before other parts of the mathematical building up, and this could be used as an argument in favour of a step-by-step approach in mathematics education.

So it would seem – as already outlined in section 2 – that an orderly presentation of geometry along the lines of the Euclidian construction would fit the Platonist conception. And indeed these are some of the guidelines that have been brought down through the generations during which the Euclidean paradigm has had such an immense influence. This actually represents how much mathematical practice has been exercised since the time of Plato. The many new developments in mathematics in modern times have of course changed the content of mathematics education. But it is nonetheless important to remember that the ruler and the pair of compasses were

still the main components in the mathematics curricula until just a few decades ago – at least in the Danish curricula. It would appear that they were there because of the tradition set by Euclid. Apparently, they figured in Euclid's *Elements* in the first place because they could produce what was thought to be the heavenly forms of infinitude, namely the straight line and the circle. The second and the third axioms of Euclidean geometry exactly points out that these are the constructions that are allowed in geometry (see the endnote 3).

The Wittgensteinian would not oppose the use of Euclidean geometry in mathematics curricula. But they would denounce the idea that the content is anything to do with heavenly spheres and the like. They would also reject any notion that this was the only and necessary way of doing mathematics, since there are so many other ways mathematics might have turned out. In this way the Wittgensteinian opens the door to thinking about topics like ethno-mathematics, or the possibility in general that mathematics is not something universal – there is no such thing as 'the' mathematics needed. It is perfectly conceivable that different cultures might find that different mathematical curricula suited them and their way of living, whereas others might even do without mathematics in the traditional sense. Wittgenstein's philosophy does not really offer much guidance in this respect, in that it does not point to a particular mathematical structure as the essence of mathematics. Rather Wittgenstein tends to suggest that there is no essence to be found, as explained earlier on.

Through his claim about the absence of any sacred 'essence' in mathematics, he is naturally saying a great deal about how we might think of the planning of mathematics education. The learning of mathematics is easily divorced from a purely abstract entity of mathematical propositions in the Wittgensteinian conception. It might not be functional analysis in complete abstraction in which university students should be trained. It could be a completely different field of mathematical investigation, perhaps connected to a highly complex practical situation where mathematics is used in a cross disciplinary setting?

Applications of mathematics are in many educational traditions divorced from the teaching of the mathematical system itself. This altogether fits in with the ideas of Platonism. Plato himself disputed his contemporary mathematicians for being too occupied with the practical uses of mathematics instead of pure reasoning.

They have in view practice only, and are always speaking in a narrow and ridiculous manner, of squaring and extending and applying and the like --they confuse the necessities of geometry with those of daily life; whereas knowledge is the real object of the whole science. (Plato, *The Republic*, Book VII)

Plato is obviously not happy with the geometers of his time. They seem to him to be too engaged with the problems of everyday life and they even lower themselves to apply mathematics!

In the Wittgensteinian conception, mathematics should be thought of as a toolbox, a network of techniques for representation and measure. There is nothing to disqualify the use of practical settings for learning mathematics – rather it seems to be a natural component of any mathematics curricula.

CONCLUSIONS

We have discussed how Wittgenstein's conception of mathematics is a thoroughly social conception, in the sense that it refuses any rationalistic order of reasoning or world of mathematical objects of any sort to enter into our description of the nature of mathematics. In this way he presents us with a vision for doing research in mathematics education that represents the learning processes of mathematics as purely social. I have argued that his philosophical perspective favours a sociological foundation for doing research in mathematics education.

In addition to this we have discussed how Wittgenstein establishes a novel understanding of what it means for mathematics to be a social construction. In recent years approaches called 'Humanist' or 'Social' mathematics have developed, that in a sense have the same agenda as Wittgenstein, namely to effectively show how mathematics is a social construction. But the social constructivist schools have focused on an epistemological critique of former 'absolutist' schools in the philosophy of mathematics. They consider mathematics to be essentially subject to historical changes and mathematical knowledge to be fallible, not absolute (Hersh, 1997, p. 22). They think, in accordance with Lakatos, that it is impossible to find a foundation of mathematics that can secure mathematical knowledge.

Despite the vast difference between the Platonist and the Wittgensteinian conceptions, they may actually be said to agree that mathematics is more absolute than fallible – but for completely different reasons. For the Platonist mathematics is absolute because it is out there before we get to it, while to the Wittgensteinian, it is absolute because it serves the same purpose as the meter in Paris – a measure that is the result of a socio-historical development over many centuries. Questioning whether it is true or not is nonsensical because it is precisely a measure. Despite their basic ideas often overlapping, Wittgenstein therefore differs from the social constructivists in that he does bring another perspective to the nature of mathematics; one that excludes any possibility of talking about the fallibility of mathematics. Instead he points to language-philosophical conclusions that certainly seem worthwhile in establishing an opposition to many Platonist elements in our thinking about mathematics.

NOTES

1. See The Prime Pages – Prime number research, records and resources for elaboration: (<http://primes.utm.edu>).

2. Plato's academy existed for no less than 900 years before the Roman emperor Justinian abolished it in 529 AD.

3. The five axioms in a modern version look like this: 1) A straight line segment can be drawn joining any two points. 2) Any straight line segment can be extended indefinitely in a straight line. 3) Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center. 4) All right angles are congruent. 5) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate. (Source: Weisstein, Eric W. "Euclid's Postulates." From MathWorld--A Wolfram Web Resource. <http://mathworld.wolfram.com/EuclidsPostulates.html>).

4. Shanker has called it 'a turning point in the philosophy of mathematics' in his extensive book on the subject (Shanker, 1987).

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